Solving polynomial equations in $\mathbb{C}$ (section 1.1)

1) Every non-zero complex number z_0 has two square roots, i.e solutions z to

$$z^2 = z_0$$

and they are opposites.

A. $z = \sqrt{\rho}\, e^{i\phi/2}$

or

$z = \sqrt{\rho}\, e^{i(\phi/2+\pi)}$

$= -\sqrt{\rho}\, e^{i\phi/2}$

bcs $e^{i\pi} = -1$.

proof 1: Use the polar form, writing $z_0 = \rho\, e^{i\phi}$, $z = r\, e^{i\theta}$, with $r, \rho > 0$.

$|z|\, e^{i \arg z}$

$$(re^{i\theta})^2 = \rho e^{i\phi}$$

$$r^2 e^{i(2\theta)} = \rho e^{i\phi}$$

$|\ |$: $r^2 = \rho$

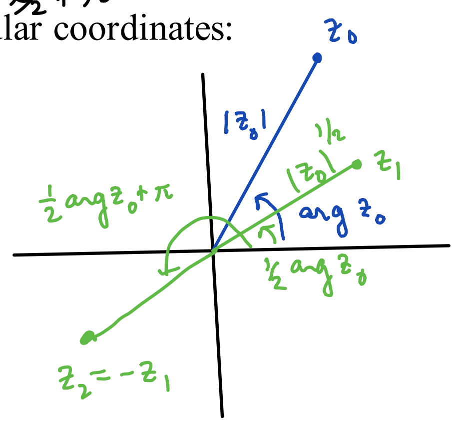
i.e. $r = |z| = \sqrt{|z_0|}$

args: $2\theta = \phi + 2\pi k \quad k \in \mathbb{Z}$

$\theta = \phi/2 + \pi k$

$\theta = \phi/2,\ \phi/2 + \pi$

proof 2: (To convince you how great polar form is) Use rectangular coordinates: Express z_0, z in terms of their real and imaginary parts,



$$z_0 = x_0 + i\, y_0$$
$$z = x + i\, y$$
$$(x + i\, y)^2 = x_0 + i\, y_0$$
$$\begin{cases} x^2 - y^2 = x_0 \\ 2\, x\, y = y_0 \end{cases}$$

Case 1: If $y_0 \neq 0$ then $x, y \neq 0$. Solve for y from the second equation and subsitute into the first:

$$x^2 - \left(\frac{y_0}{2\, x}\right)^2 = x_0$$
$$4\, x^4 - 4\, x_0 x^2 - y_0^2 = 0.$$

Use the quadratic formula for real coefficients for x^2

$$x^2 = \frac{4\, x_0 + \sqrt{16\, x_0^2 + 16\, y_0^2}}{8} = \frac{x_0 + \sqrt{x_0^2 + y_0^2}}{2}.$$

There are two opposite real values of x which solve this equation, with corresponding opposite values of $y = \dfrac{y_0}{2\, x}$.

Case 2: If $y_0 = 0$ it meant that $z_0 = x_0$ was real, and you already know how to find the two square roots. If $x_0 > 0$ they will be real square roots, and if $x_0 < 0$ they will be imaginary.

2) The general degree n polynomial equation

$$p(z) := z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0.$$

$$a_{n-1}, \ldots a_1, a_0 \in \mathbb{C}$$

You've been told forever that every degree n polynomial equation has n complex roots, counting multiplicty. This fact is known as "The Fundamental Theorem of Algebra." You'll learn a beautiful "elementary" proof of the fundamental theorem of algebra in this class. It's a proof by contradiction though, and except for very special polynomials there are no explicit formulas for exact solutions.....

There is a very complicated cubic formula for cubic equations. There is also a formula for the roots of 4^{th} order polynomials. The Abel-Ruffini Theorem asserts however, that there are no general formulas for the roots of degree 5 and higher polynomial equations, such that these formulas use only the algebraic operations of addition, multiplication, and taking radicals (square roots, cube roots, etc.). One of the founders of number theory, Évariste Galois, developed "Galois Theory", which explains exactly which special higher degree polynomial equations can be solved using these operations. These are topics in advanced algebra courses.

3) The special polynomial equation $z^n = 1$. $n = 1, 2, 3, \ldots$

Its solutions are called "the n^{th} roots of unity", and there are n of them.

Since complex multiplication in polar form reads

$$z\, w = r\, e^{i\,\theta}\, \rho\, e^{i\,\phi} = r\, \rho\, e^{i\cdot(\theta + \phi)},$$

(where $r = |z|$, $\theta = arg(z)$, $\rho = |w|$, $\phi = arg(w)$), it's easy to check via induction, that

$$\boxed{z^n = r^n\, e^{i\,n\,\theta}}\, .$$

This formula for powers is known as "DeMoivre's formula".

induction If $z = re^{i\theta}$

$n=1$: $z = re^{i\theta}$ ✓

assume $n=k$: $z^k = r^k e^{ik\theta}$

$$\Rightarrow z^{k+1} = z^k z = (r^k e^{ik\theta})(re^{i\theta}) \overset{\text{mult prop}}{=} r^{k+1} e^{i(k+1)\theta}$$

So, to solve $z^n = 1$, express $z = |z|\, e^{i\,\theta}$ and solve

$$z^n = |z|^n\, e^{i\,n\,\theta} = 1. = 1e^{i0}$$

$|\ |$: $|z|^n = 1 \Rightarrow |z| = 1$ $z \in S^1$, the unit circle

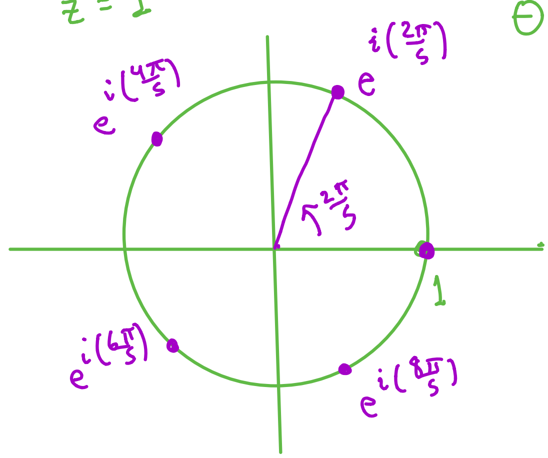
arg: $n\theta = 0 + 2\pi k$ $k \in \mathbb{Z}$

$\theta = \frac{2\pi}{n} k$

$\theta = 0, \frac{2\pi}{n}, \frac{2\pi}{n}\cdot 2, \ldots \frac{2\pi}{n}(n-1), \cancel{\frac{2\pi}{n} n} \equiv 0$

$n-1$ diff pts.

$z^5 = 1$

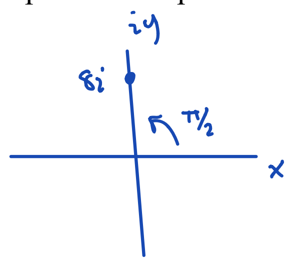


Name Solutions **Student I.D.**

Math 4200-001 Quiz week 1 August 26, 2020

1a) Express $8\,i$ in polar form, using Euler's formula

(2 points)



$|8i| = 8$

$\arg 8i = \pi/2 \quad (+2\pi k)$

$$\boxed{8i = 8e^{i\pi/2}}$$

1b) Find all sollutions z to the equation $z^3 = 8\,i$.

(6 points)

$z = |z|e^{i\theta}$

$z^3 = |z|^3 e^{i3\theta} = 8e^{i\pi/2}$

$|\ |: \quad |z|^3 = 8 \Rightarrow |z| = 2$

$\arg: \quad 3\theta = \frac{\pi}{2} + 2\pi k, \; k \in \mathbb{Z}$

$\theta = \frac{\pi}{6} + \frac{2}{3}\pi k = \frac{\pi}{6} + \frac{4\pi}{6}k$

$\theta = \frac{\pi}{6}, \frac{5}{6}\pi, \frac{9}{6}\pi$ or equiv.

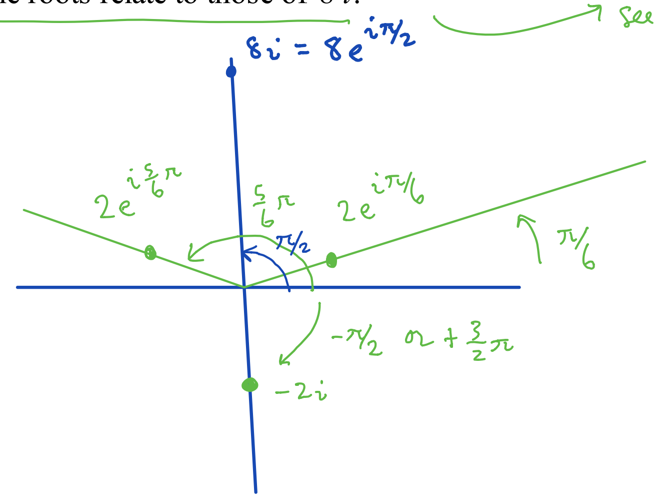
$z = 2e^{i\pi/6}, \; 2e^{i\frac{5\pi}{6}}, \; \underbrace{2e^{i\frac{3}{2}\pi}}_{(=-2i)}$

you could also solve/express sol'tn in rectangular coords

1c) Sketch $8\,i$ and the three cube roots from part b, in the complex plane. Indicate how the absolute value and arguments of the roots relate to those of $8\,i$.

see above, & picture

(2 points)



Math 4200
Friday August 28
1.2-1.3 Algebra and geometry of complex arithmetic from Wednesday's notes; introduction to complex plane transformations section 1.3, in today's notes. We'll pick up in Wednesday's notes where we left off, and continue into today's.

Announcements: Office Hrs Tuesdays 2-3
(already MWF 1-2.)

Warm-up exercise:

Section 1.3, "basic" complex functions.

Example 1 $\quad f:\mathbb{C}\to\mathbb{C},\ f(\boldsymbol{z}) = a\,\boldsymbol{z} + b \quad (a, b \in \mathbb{C})$.

Using polar form makes this transformation easy to understand geometrically. Write

$$\begin{cases} \boldsymbol{z} = |\,z\,|\, e^{i\ arg\ z}, \\ a = |\,a\,|\, e^{i\ arg\ a} \end{cases}.$$

and compute $f(z)$ in order to interpret it as the composition of a rotation, a scaling, and then a translation.

$$f(z) = az + b$$

$$f(z) = |a|\underbrace{e^{i\arg a}\,|z|\,e^{i\arg z}}_{|z|\,e^{i(\arg z + \arg a)}} + b$$

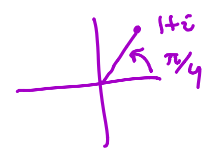
$f_1(z) = e^{i\arg a} z$ rotates z by angle a

$f_2(w) = |a| w$ scales w by $|a|$

$f_3(4) = 4 + b$ is a translation

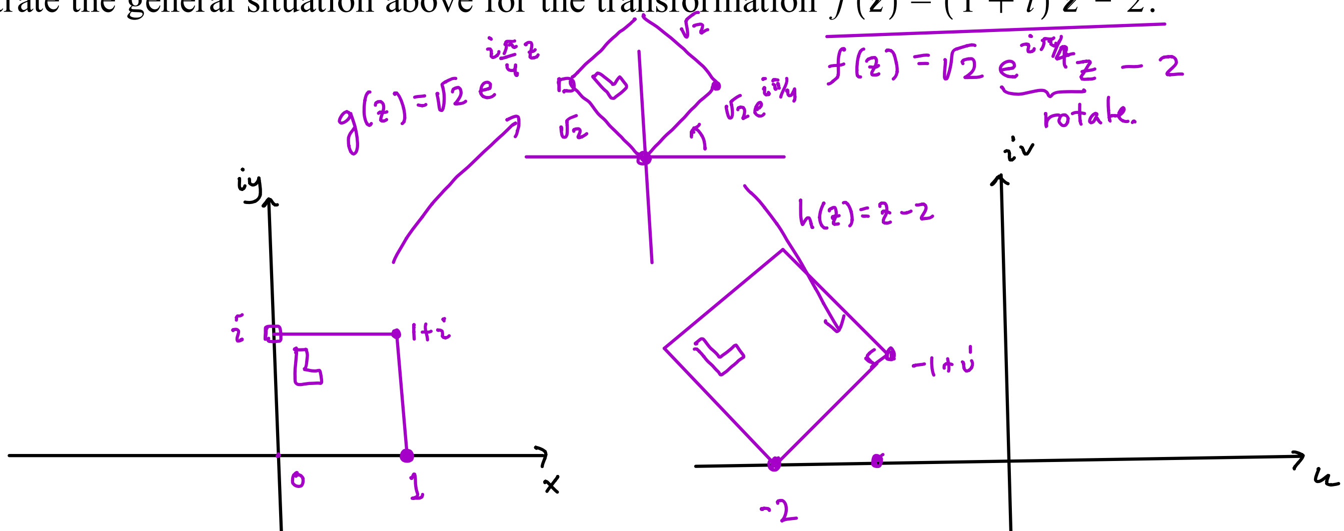
$$f = f_3 \circ f_2 \circ f_1$$

$\sqrt{2}e^{i\pi/4}$



Illustrate the general situation above for the transformation $f(\boldsymbol{z}) = (1+i)\,\boldsymbol{z} - 2$.

$f(z) = \sqrt{2}\,\underbrace{e^{i\pi/4}}_{\text{rotate.}} z - 2$



$$f(z) = h \circ g(z)$$

$$f(z) = (1+i)z - 2$$

$f(0) = -2$
$f(1) = 1 + i - 2 = -1 + i$ ✓

When we start discussing complex differentiability in section 1.5 we will want to go back and forth between discussions for complex-valued functions $f:\mathbb{C}\rightarrow\mathbb{C}$ and the mathematically equivalent discussions for related real functions $F:\mathbb{R}^2\rightarrow\mathbb{R}^2$, so that we can take advantage of what you know about continuity and differentiability for real vector-valued functions, Math 3220.

The correspondence is that for each

$$f:\mathbb{C}\rightarrow\mathbb{C}$$
$$f(z)=f(x+i\,y)=u(x,y)+i\,v(x,y)$$

with $u(x,y)=\mathrm{Re}(f(z))$, $v(x,y)=\mathrm{Im}(f(z)$, there is

$$F:\mathbb{R}^2\rightarrow\mathbb{R}^2$$
$$F(x,y)=(u(x,y),v(x,y)).$$

we sometimes write $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$

And for each $F:\mathbb{R}^2\rightarrow\mathbb{R}^2$ there is a corresponding $f:\mathbb{C}\rightarrow\mathbb{C}$.

If we apply this correspondence to the example on the previous page, with $f(z)=a\,z+b$ and

$$z=x+i\,y$$
$$a=a_1+i\,a_2$$
$$b=b_1+i\,b_2,$$

then

$$f(x+i\,y)=(a_1+i\,a_2)(x+i\,y)+(b_1+i\,b_2)$$
$$=\underbrace{(a_1\,x-a_2\,y+b_1)}+i\underbrace{(a_2\,x+a_1y+b_2)}$$

If we write the corresponding real and imaginary components of F in column form we recognize certain affine transformations from linear algebra:

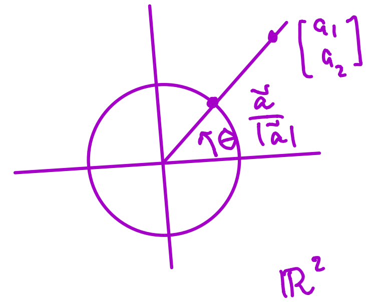
$$\vec{F}(x,y)=\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}=\begin{bmatrix} a_1\,x-a_2\,y+b_1 \\ a_2\,x+a_1y+b_2 \end{bmatrix}=\underbrace{\begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Recall your geometric matrix transformations of the plane from your linear algebra class to describe F geometrically on the next page, which will be equivalent to how we described f on the previous page.

$$\begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix} \qquad Rot_\theta: \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$=\sqrt{a_1^2+a_2^2}\begin{bmatrix} \frac{a_1}{\sqrt{\ }} & -\frac{a_2}{\sqrt{\ }} \\ \frac{a_2}{\sqrt{\ }} & \frac{a_1}{\sqrt{\ }} \end{bmatrix} \quad // \quad \uparrow Rot_\theta(\vec{e}_1)$$

pt. on unit circle



$$f(\mathbf{z}) = a\,z + b$$
$$f(\mathbf{z}) = |\,a\,|\,e^{i \arg a}\,|\,\mathbf{z}\,|\,e^{i \arg z} + b$$

$$f(x + i\,y) = (a_1 + i\,a_2)(x + i\,y) + (b_1 + i\,b_2)$$
$$= (a_1\,x - a_2\,y + b_1) + i(a_2\,x + a_1 y + b_2)$$

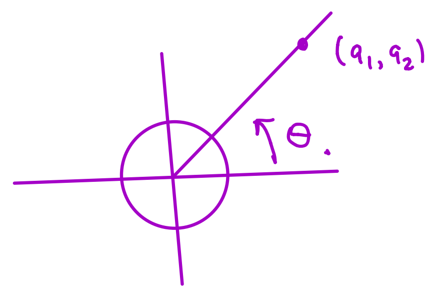
$$F(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} a_1\,x - a_2\,y + b_1 \\ a_2\,x + a_1 y + b_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}_{\sqrt{a_1^2 + a_2^2}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$+ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

translate by $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

scale by $\sqrt{a_1^2 + a_2^2}$

$$F(x,y) = F_3 \circ F_2 \circ \mathrm{Rot}_\theta \begin{bmatrix} x \\ y \end{bmatrix}$$



completely equiv descrip to

$f(z) = az + b$

$f : \mathbb{C} \to \mathbb{C}$.

interpreted for $F : \mathbb{R}^2 \to \mathbb{R}^2$